Combinatorics in Banach space theory

PROBLEMS (Part 2)*

● **PROBLEM 2.1.** Show that every almost disjoint family of subsets of ω is contained in a maximal (with respect to inclusion) almost disjoint family of subsets of ω and every such maximal family must be uncountable.

Remark. Of course, this type of argument proves only the existence of almost disjoint families of cardinality ω_1. However, we know that there exist such families of cardinality $\mathfrak{c}$.

● **PROBLEM 2.2.** Recall that a family $\mathcal{F} \subset \mathcal{P}(\omega)$ of subsets of ω is called an *independent* family whenever for every pairwise different $A_1, \ldots, A_m, B_1, \ldots, B_n \in \mathcal{F}$ the set

$$\bigcap_{i=1}^{m} A_i \cap \bigcap_{j=1}^{n} (\omega \setminus B_j)$$

is infinite. Reconstruct Hausdorff's proof of the fact that there exists an independent family of subsets of ω which has cardinality $\mathfrak{c}$. The plan is the following: Let $\mathcal{I} = \{(n, A) \colon n \in \omega,\ A \subseteq \mathcal{P}(n)\}$ and for each $X \subseteq \omega$ let $X' = \{(n, A) \in \mathcal{I} \colon X \cap n \in A\}$ (we treat natural numbers as ordinals, i.e. $n = \{0, 1, \ldots, n-1\}$). Verify that $\{X' \colon X \in \mathcal{P}(\omega)\}$ is the desired independent family.

Remark. The existence of independent families $\mathcal{F} \subset \mathcal{P}(\omega)$ with $|\mathcal{F}| = \mathfrak{c}$ was first proved by G.M. Fichtenholz and L.V. Kantorovich in 1935. F. Hausdorff's proof was published one year later. Its advantage is that it generalises to higher cardinals.

● **PROBLEM 2.3.** Let $x \in \ell_\infty/c_0$ and let $(x_n)_{n=1}^\infty \in \ell_\infty$ be any element of the coset x. Show that the norm of x in ℓ_∞/c_0 (that is, the distance from $(x_n)_{n=1}^\infty$ to the subspace c_0 of ℓ_∞) may be calculated as $\|x\| = \limsup_n |x_n|$. Use this norm to define a Lipschitz retraction of ℓ_∞ onto c_0, i.e. a surjective Lipschitz map $f \colon \ell_\infty \to c_0$ such that $f|_{c_0}$ is the identity on c_0.

Remark. The desired retraction has no chance to be linear as we know that c_0 is not complemented in ℓ_∞.

● **PROBLEM 2.4.** Show that ℓ_∞/c_0 contains a subspace isometric to $c_0(\mathfrak{c})$.

Hint. Take an almost disjoint family of subsets of ω which has cardinality $\mathfrak{c}$ and use the formula for the norm in ℓ_∞/c_0 (see Problem 2.3).

● **PROBLEM 2.5.** Show that $\ell_1(\Gamma)$ has the Schur property, for any non-empty set Γ.

Hint. You are welcomed to use the Schur property of ℓ_1.

● **PROBLEM 2.6.** Let X be an infinite-dimensional closed subspace of ℓ_1. Prove that X^* is non-separable.

Hint. In any infinite-dimensional normed space the weak closure of the unit sphere is the unit ball (why?); in particular $0 \in \overline{S_X}^w$.

*Evaluation: ●=2pt, ●=3pt, ●=4pt

● **PROBLEM 2.7.** By using the Fichtenholz–Kantorovich–Hausdorff theorem (which is to be proved in Problem 2.9) show that for every infinite cardinal number κ the space $\ell_1(2^\kappa)$ is isometrically isomorphic to a subspace of $\ell_\infty(\kappa)$. This generalises Problem 1.14(b).

Hint. This is really easy with such a heavy tool in hand!

Remark. With the same (short) proof one may show that if K is a compact Hausdorff space admitting an independent family of clopen sets of cardinality λ, then $\ell_1(\lambda)$ embeds isometrically into $C(K)$. The Stone–Čech compactification $\beta\kappa$ of any infinite cardinal number κ (equipped with the discrete topology) admits a κ-independent family of clopen sets of cardinality 2^κ, which follows directly from the Fichtenholz–Kantorovich–Hausdorff theorem and the fact that $\beta\kappa$ is nothing else but the Stone space for the Boolean algebra $\mathcal{P}(\kappa)$. Moreover, we know that $C(\beta\kappa) \simeq \ell_\infty(\kappa)$ isometrically, so the above statement yields a more universal version of what is claimed in the problem.

● **PROBLEM 2.8.** Let K be a compact Hausdorff space for which there is a disjoint family of cardinality κ consisting of non-empty open sets. Show that $c_0(\kappa)$ is isometrically isomorphic to a subspace of $C(K)$.

● **PROBLEM 2.9.** Let κ be an infinite cardinal number. A family $\mathcal{F} \subset \mathcal{P}(\kappa)$ of subsets of κ is called a *κ-independent* family whenever for every pairwise different $A_1, \ldots, A_m$, $B_1, \ldots, B_n \in \mathcal{F}$ the set

$$\bigcap_{i=1}^{m} A_i \cap \bigcap_{j=1}^{n} (\kappa \setminus B_j)$$

has cardinality κ. Generalise Hausdorff's proof from Problem 2.2 to show that there exists a κ independent family of subsets of κ which has cardinality 2^κ.

Remark. The classical application of the above assertion is in the proof of Pospíšil's theorem from 1939, which asserts that there exist exactly 2^{2^κ} *uniform* (that is, having all elements of the same cardinality) ultrafilters on κ.

● **PROBLEM 2.10.** Prove the converse to the assertion of Problem 2.8: If κ is a cardinal number and K is a compact Hausdorff space such that $C(K)$ contains an isomorphic (not necessarily isometrically) copy of $c_0(\kappa)$, then K has a disjoint family of cardinality κ consisting of non-empty open sets.

Hint. By the assumption, there is a bounded linear operator $T\colon c_0(\kappa) \to C(K)$ which is an isomorphism onto its range, which means that T is bounded below. Consider the ranges of the standard unit vectors $(e_\alpha)_{\alpha<\kappa}$ of $c_0(\kappa)$ and, with the aid of them, define κ open subsets of K which do not intersect 'too much'. Then, try to extract an appropriate subfamily. This has a bit of combinatorial flavour, similar to that of Rosenthal's lemma.

Remark. Since $\beta\mathbb{N}$ is separable ($\mathbb{N}$ is dense in $\beta\mathbb{N}$), it satisfies the c.c.c. (the *countable chain condition*, i.e. every disjoint family of non-empty open sets is countable). Hence, because of the identification $C(\beta\mathbb{N}) \simeq \ell_\infty$, the above assertion implies that $c_0(\omega_1)$ does not embed isomorphically into ℓ_∞, which was to be proved in a more direct way in Problem 1.14(a) (the hints suggested therein were also good enough to replace $\mathfrak{c}$ by ω_1).

● **PROBLEM 2.11.** Let $p \in [1,\infty) \setminus \{2\}$ and $T\colon \ell_p \to \ell_p$ be a surjective linear isometry. Show that there exists a permutation π of the set of all natural numbers and a sequence of scalars $(\varepsilon_n)_{n=1}^\infty$ with $|\varepsilon_n| = 1$ for $n \in \mathbb{N}$ such that $T(x) = (\varepsilon_n x_{\pi(n)})_{n=1}^\infty$ for every $x = (x_n)_{n=1}^\infty \in \ell_p$.

Hint. First show that for any $x, y \in \ell_p$ the equalities $\|x+y\|^p = \|x-y\|^p = \|x\|^p + \|y\|^p$ imply that x and y have disjoint supports. For $p = 1$ it is easy; for $p > 1$ you are allowed to use the following Lamperti–Clarkson inequality: If $\varphi\colon [0,\infty) \to [0,\infty)$ is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(\sqrt{t})$ is convex, then

$$\varphi(|z+w|) + \varphi(|z-w|) \geqslant 2\varphi(|z|) + 2\varphi(|w|) \quad \text{for all } z, w \in \mathbb{C}.$$

The reverse inequality is valid provided $\varphi(\sqrt{t})$ is concave, and in the case where that convexity or concavity is strict the above inequality becomes equality if and only if $zw = 0$ (see [R.J. Fleming, J.E. Jamison, *Isometries on Banach Spaces*, vol. 1, Lemma 3.2.1]). Then, use that observation for $x = T(e_m)$ and $y = T(e_n)$, where e_m and e_n are standard unit vectors in ℓ_p with $m \neq n$.

Remark. Recall that the famous theorem due to S. Mazur and S. Ulam (1932) says that any *surjective* isometry between two *real* normed spaces must be affine (that is, a translation of a linear map). Combining this with the above assertion we get the general description of any surjective isometry from the real ℓ_p-space onto itself, for $p \in [1,\infty) \setminus \{2\}$.

● **PROBLEM 2.12.** Show that the space ℓ_2^n is not isometric to any subspace of c_0, whenever $n \geqslant 2$.

Hint. How many extreme points are there on the unit ball of ℓ_n^2? How many on the unit ball of any finite-dimensional subspace of c_0? To answer the latter question you may use Problem 1.8(a) and the fact that finite-dimensional balls are compact.

Remark. This problem should be contrasted with the (quite simple) fact that every Banach space X is *finitely representable* in c_0, that is, for every finite-dimensional subspace E of X, and every $\varepsilon > 0$, there exists a finite-dimensional subspace F of c_0 with $\dim E = \dim F$ and a linear isomorphism $T\colon E \to F$ satisfying $\|T\| \cdot \|T^{-1}\| < \varepsilon$ (see [F. Albiac, N.J. Kalton, *Topics in Banach Space Theory*, §11.1]). In particular, ℓ_2 is finitely representable in c_0 which means that the finite-dimensional spaces ℓ_2^n (for $n \in \mathbb{N}$) are arbitrarily close (in the sense of the Banach–Mazur distance) to some finite-dimensional subspaces of c_0. However, for $n \geqslant 2$ they are never isometric to them.

Let us mention that A. Dvoretzky proved in 1961 that ℓ_2 is finitely representable in *every* Banach space. This is one of the deepest and most fundamental achievements in geometry of Banach spaces.

● **PROBLEM 2.13.** Let X be a Banach space. Suppose $T\colon \ell_\infty \to X$ is a weakly compact linear operator which vanishes on the subspace c_0. Prove that there exists an infinite set $A \subset \mathbb{N}$ such that T vanishes on $\ell_\infty(A)$, where

$$\ell_\infty(A) = \Big\{(\xi_n)_{n=1}^\infty \in \ell_\infty \colon \xi_n = 0 \text{ for every } n \notin A\Big\}.$$

Hint. You should use the fact that every weakly compact operator on ℓ_∞ is Dunford–Pettis (see Problem 1.11). Try also to mimic the argument in the proof of Proposition 1.4 from the lecture notes.

Remark. A Banach space Y is said to have the *Dunford–Pettis property* if every weakly compact operator from Y into another Banach space is a Dunford–Pettis operator. Examples of such spaces are: $C(K)$-spaces with K being a compact and Hausdorff space (A. Grothendieck, 1953), so in particular the space $\ell_\infty \simeq C(\beta\mathbb{N})$, and $L_1(\mu)$-spaces with a σ-finite measure μ (N. Dunford, B.J. Pettis and R.S. Phillips, 1940); see [F. Albiac, N.J. Kalton, *Topics in Banach Space Theory*, Theorem 5.4.5].

● **PROBLEM 2.14.** Let K be a compact Hausdorff space such that $C(K)$ contains an isomorphic copy of ℓ_∞. Prove that K has a subset homeomorphic to $\beta\mathbb{N}$.

Hint. First show that the assertion follows from the following proposition: If ℓ_∞ embeds isomorphically into $C(K)$, then there exists an infinite set $L \subset K$ such that $\overline{A} \cap \overline{B} = \varnothing$ for every $A, B \subset L$ with $A \cap B = \varnothing$. To prove this statement use the adjoint operator of an isomorphism from ℓ_∞ into $C(K)$ and apply Rosenthal's lemma.